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On the Theory of Linear Difference Equations.[*]

BY R. D. CARMICHAEL.

Under the above general heading I present here a discussion of each of the following topics:

I. On the Finite Integral of a Function.
II. Entire Functions Defined by Linear Homogeneous Equations with Rational Coefficients.
III. Applications of the Preceding Results to Non-homogeneous Equations.

The three parts of the paper are closely connected by having the common purpose of investigating the character of functions defined by linear difference equations, especial emphasis being put on the possibility of obtaining fundamental systems of solutions each function of which is an entire function. Where such a fundamental system does not exist, solutions are obtained having as simple sets of singularities as possible.

Each part of the paper has its own introduction, containing references to the pertinent literature and giving a brief statement of the results obtained in that part.

I. ON THE FINITE INTEGRAL OF A FUNCTION.

Introduction.

Guichard[†] has proved that if $G(x)$ is any entire function there exists another entire function $u(x)$ satisfying the relation

$$u(x+1)-u(x)=G(x).$$

Such a function $u(x)$ he obtains in the form of an integral. Appell[‡] and Hurwitz[§] have given other proofs of the same theorem. Hurwitz,[||] by an elegant method, has proved the additional theorem that if $\phi(x)$ is any function of x which is meromorphic throughout the finite plane there exists a function

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[*] Read before the American Mathematical Society, Chicago, April 1912.

[†] *Annales Scientifiques de l'École Normale Supérieure,* (3) 4 (1887), pp. 361–380.

[‡] (Liouville) *Journal de Mathématiques* (4) 7 (1891), pp. 157–219; especially pp. 159–176.

[§] *Acta Mathematica* 20 (1897), pp. 285–312.

[||] *Loc. cit.,* pp. 308–312.

$F(x)$ which is also meromorphic throughout the finite plane and satisfies the relation

$$F(x+1)-F(x)=\phi(x).$$

The primary object of the present part of this paper is to prove Guichard's important theorem by direct means. Accordingly, in § 1 a power series for $u(x)$ is assumed and direct substitution is made in the first equation above. There results an infinite system of equations for the determination of the coefficients in the series. This system is solved by what is believed to be a method of considerable value,[*] the solution obtained being such that the power series converges throughout the finite plane.

In § 2 I determine properties of the finite integral of a function which is analytic outside of a given circle, except possibly at infinity.

In §§ 3 and 4 I treat the same problem for functions which are analytic within a circle and within a circular ring respectively.

§ 1. *The Finite Integral of an Entire Function.*

Let $G(x)$ be any entire function and let it be expanded in the form

$$G(x)=a_0+a_1x+\frac{a_2}{2!}x^2+\frac{a_3}{3!}x^3+\ldots.$$

We shall show that there exists an entire function $u(x)$ satisfying the relation

$$u(x+1)-u(x)=G(x). \tag{1}$$

Let us write

$$u(x)=u_0+u_1x+\frac{u_2}{2!}x^2+\frac{u_3}{3!}x^3+\ldots,$$

and substitute in (1). If we equate the coefficients of like powers of x we have the following infinite system of equations in $u_1, u_2, \ldots,$

$$\sum_{j=1}^{\infty}\frac{1}{j!}u_{i+j}=a_i, \qquad i=0,1,2,\ldots. \tag{2}$$

It is clear that u_0 is arbitrary.

It is natural to expect that a solution of system (2) is linear in the a's. Accordingly, let us seek a solution in the form

$$u_k=\sum_{l=0}^{\infty}\alpha_{kl}a_l, \qquad k=1,2,\ldots. \tag{3}$$

If we substitute the value of u_k from (3) into (2) we have

$$\sum_{j=1}^{\infty}\frac{1}{j!}\sum_{l=0}^{\infty}\alpha_{i+j,l}a_l=a_i, \qquad i=0,1,2,\ldots.$$

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[*] It is my purpose in a future paper to develop the general consequences of this method of dealing with non-homogeneous infinite systems of equations.

We shall determine the α's so that each of these equations is a formal identity in the a's. A necessary and sufficient condition for this is that the α's satisfy the doubly infinite system of equations

$$\sum_{j=1}^{\infty} \frac{1}{j!}\alpha_{i+j,l} = \delta_{il}, \qquad \begin{cases} i = 0, 1, 2, \dots, \\ l = 0, 1, 2, \dots, \end{cases} \tag{4}$$

where δ_{il} is equal to 1 or 0 according as i is or is not equal to l.

For any fixed value of l we have the singly infinite system of equations

$$\sum_{j=1}^{\infty} \frac{1}{j!}\alpha_{i+j,l} = \delta_{il}, \qquad i = 0, 1, 2, \dots \tag{5}$$

for determining those α's which have the given second subscript l. The $(l+1)$th equation of the set has the second member 1; all the other equations of the set have the second member 0.

We shall now obtain a particular solution $\alpha_{kl} = \bar{\alpha}_{kl}$ of (5) in which $\bar{\alpha}_{kl} = 0$ when $k > l + 1$. For this case (5) reduces to the finite system (the order of equations being inverted)

$$\begin{aligned} \bar{\alpha}_{l+1,l} &= 1, \\ \bar{\alpha}_{l,l} + \frac{1}{2!}\bar{\alpha}_{l+1,l} &= 0, \\ \bar{\alpha}_{l-1,l} + \frac{1}{2!}\bar{\alpha}_{l,l} + \frac{1}{3!}\bar{\alpha}_{l+1,l} &= 0, \\ \dots\dots\dots\dots\dots\dots&\dots, \\ \bar{\alpha}_{1l} + \frac{1}{2!}\bar{\alpha}_{2l} + \dots + \frac{1}{(l+1)!}\bar{\alpha}_{l+1,l} &= 0. \end{aligned} \tag{6}$$

This system may readily be solved by determinants, but a solution more convenient for our purposes may be obtained in the form of a contour integral. If we write

$$\frac{\xi}{e^{\xi} - 1} = c_0 + c_1\xi + c_2\xi^2 + c_3\xi^3 + \dots, \tag{7}$$

expand the denominator in powers of ξ, clear of fractions and in the resulting equation equate to zero the coefficients of the different powers of ξ, we have for determining the c's the following equations:

$$\begin{aligned} c_0 &= 1, \\ c_1 + \frac{c_0}{2!} &= 0, \\ c_2 + \frac{c_1}{2!} + \frac{c_0}{3!} &= 0, \\ \dots\dots&\dots\dots \end{aligned}$$

Comparing this system with (6) we see that $\bar{\alpha}_{l+1-k,\,l} = c_k$. But from (7), by means of the Cauchy theorem on residues, we have

$$\int_C \frac{d\xi}{(e^\xi - 1)\xi^k} = 2\pi\sqrt{-1}\, c_k,$$

where the contour of integration C incloses the point zero and no other singularity of the integrand. Therefore

$$\bar{\alpha}_{l+1-k,\,l} = \frac{1}{2\pi\sqrt{-1}} \int_C \frac{d\xi}{(e^\xi - 1)\xi^k}, \quad k = 0, 1, \ldots, l; \ \bar{\alpha}_{kl} = 0, \quad k > l+1.$$

Clearly this result may be written in the simpler form

$$\bar{\alpha}_{kl} = \frac{1}{2\pi\sqrt{-1}} \int_C \frac{d\xi}{(e^\xi - 1)\xi^{l+1-k}}, \qquad k = 1, 2, 3 \ldots\ldots \tag{8}$$

If in (8) we let l run over all the numbers $0, 1, 2, \ldots$, we obtain a complete solution of (4). Substituting from (8) into (3) we have

$$u_k = \frac{1}{2\pi\sqrt{-1}} \sum_{s=0}^{\infty} a_s \int_C \frac{d\xi}{(e^\xi - 1)\xi^{s+1-k}}. \tag{9}$$

This result affords a *formal* solution of system (2). That in general it is divergent and hence does not afford an actual solution is easily shown. For we have

$$\frac{\xi}{e^\xi - 1} = \frac{1}{2\pi\sqrt{-1}} \sum_{s=0}^{\infty} \xi^s \int_C \frac{d\xi}{(e^\xi - 1)\xi^s},$$

as one sees readily from the value of c_k obtained above. Now the radius of convergence of this series is not greater than 2π, since $\xi/(e^\xi - 1)$ has a pole at $\xi = 2\pi\sqrt{-1}$. Hence if $a_t = m^t$, as in the case when $G(x) = e^{mx}$, where $m > 2\pi$, series (9) diverges at least for $k = 1$.

This (divergent) formal solution of (2) has been obtained by taking a particular solution of (4). It will now be shown that (4) has an infinitude of solutions, and that if an appropriately chosen particular solution be selected from this totality it will lead to an actual solution of (2).

If we write $\alpha_{kl} = \bar{\alpha}_{kl} + \beta_{kl}$, substitute in (4) and make use of the fact that $\bar{\alpha}_{kl}$ is a solution of (4), it is clear that we have

$$\sum_{j=1}^{\infty} \frac{1}{j!}\beta_{i+j,\,l} = 0, \qquad \begin{cases} i = 0, 1, 2, \ldots, \\ l = 0, 1, 2, \ldots\ldots \end{cases} \tag{10}$$

Thus for each value of l we have to solve a system of equations of the form

$$\sum_{j=1}^{\infty} \frac{1}{j!}\beta_{i+j} = 0, \qquad i = 0, 1, 2, \ldots\ldots \tag{11}$$

If $\beta(x)$ is any periodic function of period 1 which is analytic at zero, and we write

$$\beta(x)=\beta_0+\beta_1 x+\frac{\beta_2}{2!}x^2+\frac{\beta_3}{3!}x^3+\ldots,$$

substitute in the equation $\beta(x+1)-\beta(x)=0$ and equate to zero the coefficients of like powers of x we have system (11). Therefore, for every periodic function of period 1, and analytic at zero, there exists a corresponding formal solution of (11); and this solution is furnished by the coefficients $\beta_1, \beta_2, \ldots$ in the expansion of $\beta(x)$ above. If this expansion converges for $x=1$ the formal solution becomes an actual solution. This result will be used in constructing an appropriate solution of (10).

From the function $ce^{2n\pi\sqrt{-1}x}$, where c is any constant and n is any positive or negative integer, we have a solution $\beta_k=c(2n\pi\sqrt{-1})^k$ of (11). Now the residue of the function

$$\frac{1}{(e^\xi-1)\xi^{l-k+1}},$$

at the point $2n\pi\sqrt{-1}$, is $(2n\pi\sqrt{-1})^{k-l-1}$. Hence, if $c=(2n\pi\sqrt{-1})^{-l-1}$, we have β_k equal to the residue in consideration. That is, we have

$$\beta_k=\frac{1}{2\pi\sqrt{-1}}\int\frac{d\xi}{(e^\xi-1)\xi^{l-k+1}},$$

where the contour of integration incloses the point $2n\pi\sqrt{-1}$ and no other singularity of the integrand.

Now the sum of any finite number of solutions of system (11) is also a solution; hence it is clear that a set of values satisfying (10) is

$$\beta_{kl}=\frac{1}{2\pi\sqrt{-1}}\Sigma\int\frac{d\xi}{(e^\xi-1)\xi^{l-k+1}},\qquad \begin{cases}k=1, 2, \ldots, \\ l=0, 1, 2, \ldots,\end{cases}$$

where for a fixed l the sum of $2l$ integrals is to be taken, one around each of the $2l$ contours inclosing separately each of the $2l$ points $2n\pi\sqrt{-1}$, $n=\pm 1, \pm 2, \ldots, \pm l$ and no other singularity of the integrand. Combining this value of β_{kl} with that of $\bar{\alpha}_{kl}$ in (8) we have for α_{kl} the value

$$\alpha_{kl}=\frac{1}{2\pi\sqrt{-1}}\int_{C_l}\frac{d\xi}{(e^\xi-1)\xi^{l-k+1}},\qquad \begin{cases}k=1, 2, \ldots, \\ l=0, 1, 2, \ldots,\end{cases}$$

where the contour of integration C_l incloses the points $2n\pi\sqrt{-1}$, $n=0, \pm 1, \pm 2, \ldots, \pm l$ and no other singularity of the integrand.

Substituting this value of α_{kl} in (3) we have

$$u_k = \frac{1}{2\pi\sqrt{-1}} \sum_{s=0}^{\infty} a_s \int_{C_s} \frac{d\xi}{(e^{\xi}-1)\xi^{s-k+1}}, \qquad k = 1, 2, \ldots\ldots \tag{12}$$

In order to show that this is a valid solution of (2) we must first prove that the series is convergent for every value of k.

Denote by I_{sk} the integral

$$I_{sk} = \int_{C_s} \frac{d\xi}{(e^{\xi}-1)\xi^{s-k+1}}. \tag{13}$$

Choose as the contour of integration the square formed by four lines each at a distance $(2s+1)\pi$ from the origin, two of them being parallel to the axis of reals and the other two being parallel to the axis of imaginaries. Making the transformation $\xi = (2s+1)\pi w$, we have

$$I_{sk} = \frac{1}{(2s+1)^{s-k}\pi^{s-k}} \int_S \frac{dw}{(e^{(2s+1)\pi w}-1)w^{s-k+1}},$$

where the contour S is the square formed by four lines each at a distance unity from the origin, two of them being parallel to the axis of reals and the other two being parallel to the axis of imaginaries. Along the contour of integration $1/|e^{(2s+1)\pi w}-1|$ remains finite; let us say that it is always less than A. Along the path of integration $|w|$ is equal to or greater than 1. Hence

$$|I_{sk}| < \frac{A}{(2s+1)^{s-k}\pi^{s-k}} \int_S |dw| = \frac{8A}{(2s+1)^{s-k}\pi^{s-k}}. \tag{14}$$

From this result it follows that the series in (12) is convergent for every k provided that the same is true of the series

$$\sum_{s=0}^{\infty} \frac{a_s}{(2s+1)^{s-k}\pi^{s-k}}.$$

That this series is convergent follows at once by comparison with the convergent series

$$\sum_{s=0}^{\infty} \frac{a_s}{s!} \frac{1}{\pi^{s-k}},$$

the convergence of the latter being a consequence of the fact that $G(x)$ is entire.

Substituting these values for u_k in the expansion of $u(x)$ and choosing a convenient value for the arbitrary constant u_0, we have the following result:

$$u(x) = \frac{1}{2\pi\sqrt{-1}} \sum_{k=0}^{\infty} \frac{x^k}{k!} \sum_{s=0}^{\infty} a_s \int_{C_s} \frac{d\xi}{(e^{\xi}-1)\xi^{s-k+1}}. \tag{15}$$

We shall next determine the circle of convergence of this power series.

Let R denote any circle of radius R about the point zero. We shall show that (15) is convergent in this circle R. Form the set of functions

$$u_s(x) = \frac{1}{2\pi\sqrt{-1}} \sum_{k=0}^{\infty} \frac{x^k}{k!} a_s \int_{C_s} \frac{d\xi}{(e^{\xi}-1)\xi^{s-k+1}}, \quad s = 0, 1, 2, \ldots\ldots \tag{16}$$

In view of the inequality (14) it is easy to see that each of these series is convergent in the circle R. One has only to compare the sth series with the series

$$\sum_{k=0}^{\infty} \frac{R^k}{k!} a_s \frac{8A}{(2s+1)^{s-k}\pi^{s-k}},$$

the convergence of the latter being obvious.

Next we shall show that the series

$$u_0(x) + u_1(x) + u_2(x) + \ldots. \tag{17}$$

is uniformly convergent throughout R. By a theorem of Weierstrass it will then follow that the sum is analytic in R and may be written in the form of the second member of (15). From (14) and (16) we have

$$|u_s(x)| < \alpha_s, \quad |x| \leqq R,$$

where

$$\alpha_s = \frac{1}{2\pi} \sum_{k=0}^{\infty} \frac{R^k}{k!} |a_s| \frac{8A}{(2s+1)^{s-k}\pi^{s-k}}.$$

One shows readily that

$$\frac{\alpha_{s+1}}{\alpha_s} = \frac{|a_{s+1}|}{|a_s|} \frac{(2s+1)^s}{(2s+3)^{s+1}} \frac{1}{\pi} \frac{\sum\limits_{k=0}^{\infty} \frac{R^k}{k!}(2s+3)^k\pi^k}{\sum\limits_{k=0}^{\infty} \frac{R^k}{k!}(2s+1)^k\pi^k}$$

$$= \frac{|a_{s+1}|}{|a_s|} \frac{(2s+1)^s}{(2s+3)^{s+1}} \frac{1}{\pi} e^{2R\pi}.$$

Hence

$$\frac{\alpha_{s+1}}{\alpha_s} < \frac{|a_{s+1}|}{|a_s|} \frac{1}{2s+3} \frac{1}{\pi} e^{2R\pi} < \frac{|a_{s+1}|}{|a_s|} \frac{1}{s+1} \frac{e^{2R\pi}}{2\pi}.$$

Since $G(x)$ is an entire function the series

$$\sum_{s=0}^{\infty} |a_s| \frac{1}{s!} \bar{R}^s$$

is convergent for every value of $\bar{R}$. Hence the ratio of two consecutive terms,

$$\frac{|a_{s+1}|}{|a_s|} \frac{1}{s+1} \bar{R},$$

is ultimately less than or equal to one. If $\bar{R} > e^{2R\pi}/2\pi$ this ratio is greater than the ratio α_{s+1}/α_s; and therefore the series

$$\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \ldots .$$

is convergent, since α_{s+1}/α_s ultimately becomes and remains less than 1. But $|u_s(x)| < \alpha_s$, $|x| \leqq R$; and therefore series (16) is uniformly convergent throughout R. Hence its sum is analytic throughout R and can be written in the form (15). But R is any circle about the origin. Hence (15) converges throughout the entire plane; that is, $u(x)$ is an entire function.

It is easy to show directly that $u(x)$ as defined in (15) actually satisfies relation (1). We have thus proved the following theorem:

I. *If $G(x)$ is an entire function expansible in the form*

$$G(x) = a_0 + a_1 x + \frac{a_2}{2!}x^2 + \frac{a_3}{3!}x^3 + \ldots ,$$

then there exist entire functions $u(x)$ satisfying the difference equation

$$u(x+1) - u(x) = G(x).$$

One such entire function is

$$u(x) = \frac{1}{2\pi\sqrt{-1}} \sum_{k=0}^{\infty} \frac{x^k}{k!} \sum_{s=0}^{\infty} a_s \int_{C_s} \frac{d\xi}{(e^\xi - 1)\xi^{s-k+1}},$$

where the contour of integration C_s incloses the points $2n\pi\sqrt{-1}$, $n = 0, \pm 1, \pm 2, \ldots, \pm s$, and no other singularity of the integrand.

Evidently, the general solution of the difference equation is obtained by adding to the above solution any arbitrary periodic function of period 1. If the solution is restricted to be entire this periodic function must be entire.

The value of the integral I_{sk}, defined by equation (13), takes a very simple form when $s - k$ is even. For, then, the residues of the integrand at the two points $2n\pi\sqrt{-1}$ and $-2n\pi\sqrt{-1}$, $n \neq 0$, being respectively $(2n\pi\sqrt{-1})^{k-s-1}$ and $(-2n\pi\sqrt{-1})^{k-s-1}$, are such that the part of the integral which comes from these two residues is zero. Hence

$$I_{sk} = \int_{C_0} \frac{d\xi}{(e^\xi - 1)\xi^{s-k+1}}, \text{ when } s - k \text{ is even.}$$

That is, I_{sk} is in this case $2\pi\sqrt{-1}$ times the coefficient of ξ^{s-k+1} in the development of $\xi/(e^\xi - 1)$ in powers of ξ. Hence, when $s - k = 0$ we have $I_{sk} = -\pi\sqrt{-1}$. It is easy to see that I_{sk} is equal to zero when $s - k$ is even and not zero. To prove this it is sufficient to notice that

$$\frac{\xi}{e^\xi - 1} + \tfrac{1}{2}\xi \quad \text{or} \quad \tfrac{1}{2}\frac{\xi(e^\xi + 1)}{e^\xi - 1}$$

is an even function; for then the coefficients of odd powers of ξ (greater than the first) in the development of $\xi/(e^\xi - 1)$ are all zero.

Making use of these results as to the value of I_{sk} we have at once the following theorem:

II. *The function* $u(x)$ *of Theorem I may be written in the form*

$$u(x) = -\tfrac{1}{2}G(x) + \frac{1}{2\pi\sqrt{-1}} \sum_{k=0}^{\infty} \frac{x^k}{k!} \sum_{s=0}^{\infty}{}' a_s \int_{C_s} \frac{d\xi}{(e^\xi - 1)\xi^{s-k+1}},$$

where the accent on the second summation sign indicates that for each value of k *only those values of* s *are to be taken for which* $s - k$ *is odd.*

COROLLARY 1. *If* $G(x)$ *is an even function* (so that a_s is zero when s is odd), *then the equation*

$$u(x+1) - u(x) = G(x)$$

has a solution which is the sum of $-\tfrac{1}{2}G(x)$ *and an odd function which is entire.*

COROLLARY 2. *If* $G(x)$ *is an odd function* (so that a_s is zero when s is even), *then the equation*

$$u(x+1) - u(x) = G(x)$$

has a solution which is the sum of $-\tfrac{1}{2}G(x)$ *and an even function which is entire.*

§ 2. *The Finite Integral of a Function which is Analytic Outside of a Circle, except at Infinity.*

Let us consider the equation

$$u(x) = G(x) + \frac{c}{x-a} + \phi(x) \equiv X(x), \tag{18}$$

where $G(x)$ is an entire function and $\phi(x)$ is analytic outside of a circle of finite radius R about the point a as a center, the development of $\phi(x)$ in negative powers of $x - a$ containing no constant term and no term in $(x-a)^{-1}$. It is evident that a solution $u(x)$ of (18) may be written in the form

$$u(x) = u_1(x) + u_2(x) + u_3(x),$$

where $u_1(x)$, $u_2(x)$, $u_3(x)$ are solutions respectively of the equations

$$\left.\begin{aligned} u_1(x+1) - u_1(x) &= G(x)\,, \\ u_2(x+1) - u_2(x) &= \frac{c}{x-a}, \\ u_3(x+1) - u_3(x) &= \phi(x)\,. \end{aligned}\right\} \tag{19}$$

A solution $u_1(x)$ of the first equation (19), analytic throughout the finite plane, was found in the preceding section. For the second and third equations we have the following two formal solutions:

$$u_2(x) = -\frac{1}{x-a} + \sum_{i=1}^{\infty}\left(\frac{1}{i} - \frac{1}{x-a+i}\right), \quad u_2(x) = \sum_{i=1}^{\infty}\left(\frac{1}{x-a-i} + \frac{1}{i}\right),$$

$$u_3(x) = -\sum_{i=0}^{\infty}\phi(x+i), \qquad u_3(x) = \sum_{i=1}^{\infty}\phi(x-i).$$

Let us consider the first pair of formal solutions for $u_2(x)$ and $u_3(x)$. It is easy to see that there exists a line parallel to the axis of imaginaries to the right of which each of the series is uniformly convergent. Hence to the right of this line $u_2(x)$ and $u_3(x)$, as defined by these series, are analytic except at infinity. From this result and the fact that $u_1(x)$ is analytic throughout the finite plane it follows that $u(x)$ is analytic to the right of the same line. Now, by means of the equation

$$u(x-1) = u(x) - X(x-1),$$

the value of $u(x)$ is known at $x_0 - 1$ when it is known at x_0. In this way the solution obtained above may be defined throughout any finite region of the plane. Thus we see that $u(x)$ is analytic throughout the finite plane except at the singularities of $X(x)$ in the finite plane and the points congruent to them on the left. (A point $\alpha - n$ is said to be congruent to α on the left if n is a positive integer; similarly, we say that $\alpha + n$ is congruent to α on the right if n is a positive integer.)

In a similar way we may treat the second pair of formal solutions for $u_2(x)$ and $u_3(x)$. Here we consider first the part of the plane which is to the left of a suitably chosen line parallel to the axis of imaginaries. It is clear that the solution $u(x)$ of (18), which we obtain in this case, is analytic throughout the finite plane except at the points congruent on the right to the singularities of $X(x)$ in the finite plane.

Thus we have the following theorem:

III. *Let $X(x)$ be any function of x which is analytic outside of a circle about the center a of finite radius R, except possibly at infinity. Then there exists a function $u(x)$ satisfying the relation*

$$u(x+1) - u(x) = X(x)$$

and having either one of the following two properties:

1. *It is analytic throughout the finite plane except at the singularities of $X(x)$ in the finite plane and the points congruent to them on the left.*

2. *It is analytic throughout the finite plane except at points congruent on the right to the singularities of $X(x)$ in the finite plane.*

§ 3. *The Finite Integral of a Function which is Analytic within a Circle.*

Let $\psi(x)$ be any function which is analytic within a circle R of radius R about the point a as a center. A method due to Weber[*] gives us in this case an immediate solution of the problem of finding a function $u(x)$ which satisfies the equation

$$u(x+1)-u(x)=\psi(x)$$

and is analytic in any circle within the circle R. Let $\bar{R}$ be any circle about a as a center and having a radius less than that of the circle R. Denote by $(R\bar{R})$ the circular ring between these two circles. On a line through a and parallel to the axis of imaginaries take two points A and B, lying within the circular ring $(R\bar{R})$, the imaginary coefficient of A being less than that of a and the imaginary coefficient of B being greater than that of a. Let x be any point in the circle $\bar{R}$ and denote by l the line through the points $\ldots . x-1$, x, $x+1$, $\ldots . .$

Now form the integral

$$u(x)=\int_C \frac{\psi(z)\,dz}{1-e^{2\pi i(x-z)}}, \qquad i=\sqrt{-1}, \tag{20}$$

where the path of integration C extends from A to B, lies entirely within the circle R, passes through a point P lying on the line l between $x-1$ and x and has no other point in common with the line l. Then, on account of the periodicity of the integrand considered as a function of x, it is clear that

$$u(x+1)=\int_{\bar{C}} \frac{\psi(z)\,dz}{1-e^{2\pi i(x-z)}},$$

where the path of integration is as in the preceding case except that the point P is now between x and $x+1$. The difference $u(x+1)-u(x)$ is expressible as a contour integral taken along the curve made up of $\bar{C}$ taken in a positive sense and C taken in a negative sense, the integrand being the same as above. Hence, by Cauchy's theorem, $u(x+1)-u(x)=\psi(x)$. But $u(x)$, as defined by (20), is evidently an analytic function of x at every point x in the circle R. Hence we have the following theorem:

IV. *If $\psi(x)$ is analytic within a circle of radius R about the point a and $\bar{R}$ is any circle about a with radius less than R, then there exists a function $u(x)$ which is analytic throughout $\bar{R}$ and satisfies the relation*

$$u(x+1)-u(x)=\psi(x).$$

Such a function is defined in equation (20).

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[*] *Acta Mathematica,* Vol. XXVII (1903), p. 226.

§ 4. *The Finite Integral of a Function which is Analytic in a Circular Ring.*

If $\phi(x)$ is any function which is analytic within a circular ring we may deal with the problem of solving the equation

$$u(x+1)-u(x)=\phi(x)$$

in a manner similar to that employed in the preceding section. We suppose that $\phi(x)$ is analytic within the circular ring bounded by two circles each having the center a, the radii being r and R, where $r+1<R$. Take any two quantities $\bar{r}$ and $\bar{R}$ such that

$$r<\bar{r}, \quad \bar{r}+1<\bar{R}<R,$$

and form circles about a with radii $\bar{r}$ and $\bar{R}$. Denote by r, $\bar{r}$, $\bar{R}$, R the circles having the radii r, $\bar{r}$, $\bar{R}$, R respectively. Take points A and B lying within the circular ring bounded by $\bar{R}$ and R and lying on a line through a parallel to the axis of imaginaries, A being below and B above a. Let us define a region H in the following manner: Let the line AB cut $\bar{r}$ and $\bar{R}$ in the points L, l, m, M, these points being named in the order of increasing imaginary coefficients. From l and m draw tangents to $\bar{r}$ proceeding to the left and cutting $\bar{R}$ in the points T_1 and T_2 respectively. The region H has the following boundary: The arc T_1LMT_2 of the circle $\bar{R}$, the tangent T_2m, the right-hand semi-circle ml, the tangent lT_1. Let x be a point in the region H and let λ be the line through the points $\ldots . x-1, x, x+1, \ldots . .$ Form the function

$$u(x)=\int_C \frac{\phi(z)\,dz}{1-e^{2\pi i(x-z)}}, \tag{21}$$

where the path of integration C extends from A to B, lies wholly in the ring bounded by r and R, cuts the line λ in a point P between $x-1$ and x, and has no other point in common with λ.

It is easy to show, by the method employed in the preceding section, that the function $u(x)$, as thus defined, satisfies the relation $u(x+1)-u(x)=\phi(x)$. It is evidently analytic at every point of the region H.

Now

$$u(x-1)=u(x)-\phi(x-1).$$

By means of this equation the value of $u(x)$ at a point x_0-1 can be found when its value at x_0 is known. In this way we may determine the function $u(x)$ in that part of the circular ring between r and R in which it is not already defined. It is thus clear that $u(x)$ is analytic throughout this circular ring except (possibly) at the points congruent on the left to the singularities of $\phi(x)$ in r.

The right and left half-planes, as divided by the line AB, enter unsymmetrically into the above discussion. It is clear that we may make a similar discussion of the problem in which the two half-planes exchange rôles.

Thus we have the following two-fold theorem:

V. *If* $\phi(x)$ *is any function which is analytic within a circular ring bounded by two circles about a as center and having radii r and R where* $r+1<R$, *and if any other concentric circular ring is formed with radii* $\bar{r}$ *and* $\bar{R}$, *where* $r<\bar{r}$, $\bar{r}+1<\bar{R}<R$, *then there exists a function* $u(x)$ *satisfying the relation*

$$u(x+1)-u(x)=\phi(x)$$

and having either of the following properties:

1. *It is analytic throughout the ring bounded by the circles* $\bar{r}$ *and* $\bar{R}$ *except at points congruent on the left to the singularities of* $\phi(x)$ *which lie in the circle r.*

2. *It is analytic throughout the ring bounded by the circles* $\bar{r}$ *and* $\bar{R}$ *except at points congruent on the right to the singularities of* $\phi(x)$ *which lie in the circle r.*

II. ENTIRE FUNCTIONS DEFINED BY LINEAR HOMOGENEOUS EQUATIONS WITH RATIONAL COEFFICIENTS.

Let us consider a single difference equation of the nth order

$$R_0(x)H(x+n)+R_1(x)H(x+n-1)+\ldots+R_n(x)H(x)=\phi(x),$$

where $R_0(x), R_1(x), \ldots, R_n(x)$ are rational functions. Let α be the least integer such that $x^{-i\alpha}R_i(x)/R_0(x)$ is analytic at infinity for every value of i, $i=1, 2, \ldots, n$; and let R_i be the constant term in the expansion of $x^{-i\alpha}R_i(x)/R_0(x)$ in powers of x^{-1}. We assume that the constants R_i are such that the roots $\rho_1, \rho_2, \ldots, \rho_n$ of the characteristic equation

$$\rho^n+R_1\rho^{n-1}+R_2\rho^{n-2}+\ldots+R_{n-1}\rho+R_n=0$$

are in absolute value different from each other and from zero. An equation having these properties, whether homogeneous or non-homogeneous, is here said to be of *simple character.* In what follows we confine attention for the most part to equations of simple character.

If the equation (whether of simple character or not) is homogeneous and the coefficients are constants it is easy to determine by elementary means a fundamental system of solutions, each function of which is entire. Guichard* has gone further and has shown that if the coefficients are constants and the independent term $\phi(x)$ is any entire function, then there exists a particular solution of the equation which is an entire function. In this case the general solution can be expressed in the form

$$H(x) = E(x) + p_1(x)E_1(x) + \ldots\ldots + p_n(x)E_n(x),$$

where $E(x), E_1(x), \ldots\ldots, E_n(x)$ are entire functions and $p_1(x), \ldots\ldots, p_n(x)$ are arbitrary periodic functions of period unity.

The question naturally arises as to whether corresponding theorems are true for equations in which the coefficients are no longer constants, but are restricted to be rational functions. Such is the subject of investigation in this and the following section of the present paper.

The theorem on page 133 of my dissertation, taken in connection with the remarks which follow, leads at once to a theorem which we shall need; it may be stated thus:

If

$$P_0(x)H(x+n) + P_1(x)H(x+n-1) + \ldots\ldots + P_n(x)H(x) = 0$$

is a homogeneous difference equation of simple character in which the coefficients $P_0(x), P_1(x), \ldots\ldots, P_n(x)$ *are polynomials in* x, *then*

1) *There exists a fundamental system of solutions* $H_1(x), \ldots\ldots, H_n(x)$ *each function of which is analytic throughout the finite plane except at the zeros of* $P_n(x)$ *and the points congruent to them on the left. If* x *approaches infinity in the positive direction along any line parallel to the real axis, we have*

$$\lim H_j(x)\{\Gamma(x)\}^{-\alpha} a_j^{-x} x^{-m_j} = 1, \qquad j = 1, \ldots\ldots, n,$$

where α, a_j, m_j *are appropriately determined constants;*

2) *There exists a fundamental system of solutions* $\overline{H}_1(x), \ldots\ldots, \overline{H}_n(x)$ *each function of which is analytic throughout the finite plane except at the zeros of* $P_0(x-n)$ *and the points congruent to them on the right. If* x *approaches infinity in the negative direction along any line parallel to the real axis, we have*

$$\lim \overline{H}_j(x)\{\Gamma(x)\}^{-\beta} b_j^{-x} x^{-n_j} = 1, \qquad j = 1, \ldots\ldots, n,$$

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* *Annales de l'École Normale Supérieure* (3), 4 (1887), pp. 368–370.

† *Transactions of the American Mathematical Society*, Vol. XII (1911), pp. 99–134.

where β, b_j, n_j *are appropriately determined constants and*

$$\bar{\Gamma}(x) = (1 - e^{2\pi i x})\Gamma(x), \qquad i = \sqrt{-1}.$$

From either of these fundamental systems of solutions of (1) *we may obtain* (at once and in an obvious way) *a fundamental system which consists of entire functions;* for this purpose, it is sufficient to multiply each of the given solutions by an appropriate periodic function of period 1 so that the zeros of the latter will balance off the poles of the former, thus giving a product function which has no singularity in the finite plane.

Thus we see that all functions defined by linear homogeneous difference equations of simple character can be expressed in terms of entire functions and functions which are periodic of period 1. That the entire functions defined by an equation are sometimes not the most interesting ones which the equation may define is evidenced by the importance of the gamma function. It has been found of much more use than any entire function which satisfies the same difference equation as the gamma function itself.

That non-homogeneous equations, even when the coefficients and the independent term are all polynomials, do not always possess a solution which is an entire function will be shown in part III by means of an example.

III. APPLICATIONS OF THE PRECEDING RESULTS TO NON-HOMOGENEOUS EQUATIONS.

The results of the preceding parts of the paper will now be applied to an investigation of properties of the solutions of non-homogeneous equations whose coefficients are polynomials in x but whose independent term is a meromorphic function or an entire function of x. The results, other than the negative one afforded by the following example, are stated in Theorems I and II.

That non-homogeneous difference equations in which the coefficients and the independent term are polynomials do not always possess solutions which are entire functions is shown by the equation

$$xg(x+1) - (x+1)g(x) = x,$$

the general solution of which may be written in the form

$$g(x) = x\left\{p(x) + \sum_{i=1}^{\infty}\left(\frac{1}{x+1-i} + \frac{1}{i}\right)\right\},$$

where $p(x)$ is an arbitrary periodic function of x of period 1. It is obvious that $g(x)$ has singularities in the finite plane for every choice of $p(x)$. But if $p(x)$ is any periodic function which has in the finite plane no singularity other than poles, then the corresponding $g(x)$ is analytic throughout the finite plane except for poles.

This example suggests the following theorem, which will now be proved:

I. *If*

$$P_0(x)H(x+n)+P_1(x)H(x+n-1)+\ldots\ldots+P_n(x)H(x)=\phi(x) \qquad (1)$$

is a difference equation of simple character in which the coefficients are polynomials in x, *while* $\phi(x)$ *is analytic throughout the finite plane except for poles, then its general solution* $H(x)$ *may be written in the form*

$$H(x)=M(x)+p_1(x)E_1(x)+\ldots\ldots+p_n(x)E_n(x),$$

where $E_1(x), \ldots\ldots, E_n(x)$ *are entire functions,* $M(x)$ *is analytic throughout the finite plane except for poles, and* $p_1(x), \ldots\ldots, p_n(x)$ *are arbitrary periodic functions of period* 1.

In section II we have shown that the corresponding homogeneous equation (that is, the equation obtained from (1) by writing zero for $\phi(x)$) has a fundamental system of solutions each function of which is entire. We let $E_1(x), \ldots\ldots, E_n(x)$ be the functions of this solution. It is now obvious that our theorem will have been demonstrated when we have proved the existence of a single function $M(x)$ satisfying the non-homogeneous equation (1) and possessing the required character of being analytic throughout the finite plane except for poles. We shall now prove the existence of such a function. Our method is the classic one of variation of parameters.

We seek a particular solution $\overline{H}(x)$ of (1) in the form

$$\overline{H}(x)=\alpha_1(x)E_1(x)+\alpha_2(x)E_2(x)+\ldots\ldots+\alpha_n(x)E_n(x). \qquad (2)$$

In order that $\overline{H}(x)$ shall be a solution of (1) it is sufficient that the functions $\alpha_i(x)$ satisfy the set of equations

$$\begin{aligned} &E_1(x+1)\Delta\alpha_1(x)+\ldots\ldots+E_n(x+1)\Delta\alpha_n(x)=0,\\ &E_1(x+2)\Delta\alpha_1(x)+\ldots\ldots+E_n(x+2)\Delta\alpha_n(x)=0,\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\\ &E_1(x+n-1)\Delta\alpha_1(x)+\ldots\ldots+E_n(x+n-1)\Delta\alpha_n(x)=0,\\ &E_1(x+n)\Delta\alpha_1(x)+\ldots\ldots+E_n(x+n)\Delta\alpha_n(x)=\phi(x). \end{aligned} \qquad (3)$$

This may readily be shown by the method usually employed in finding a solution by means of variation of parameters. It may easily be verified directly; for, from (2) and the first equation in (3) one has

$$\overline{H}(x+1)=\alpha_1(x)E_1(x+1)+\ldots\ldots+\alpha_n(x)E_n(x+1).$$

By means of this equation and the second one in (3) one obtains the result

$$\overline{H}(x+2) = \alpha_1(x)E_1(x+2) + \ldots . + \alpha_n(x)E_n(x+2).$$

The process may evidently be continued till $\overline{H}(x+n)$ is found. Then a direct substitution in (1) will show that that equation is satisfied.

The determinant of (3) is not identically zero; for this is the condition that $E_1(x), \ldots ., E_n(x)$ form a fundamental system of solutions of the homogeneous equation corresponding to (1). Hence (3) can be solved for $\Delta\alpha_1(x), \ldots ., \Delta\alpha_n(x)$. Thus we have

$$\Delta\alpha_i(x) = M_i(x), \qquad i = 1, \ldots ., n, \tag{4}$$

where the functions $M_i(x)$, $i = 1, \ldots ., n$, are obviously analytic throughout the finite plane except for poles. From Hurwitz' theorem, quoted above (p. 163), it follows then that there exist functions $\alpha_i(x)$, $i = 1, \ldots ., n$, having the same property. If these functions $\alpha_i(x)$ are put in (2) we have a function $\overline{H}(x)$ satisfying (1) and having the property of being analytic throughout the finite plane except for poles. We let $\overline{H}(x)$ be the function $M(x)$ of the theorem. Thus the proof of the theorem is completed.

We shall now turn our attention to a more restricted form of equation (1). We suppose that $\phi(x)$ is an entire function and that $P_n(x) \equiv 1$. We write

$$P_0(x)H(x+n) + \ldots . + P_{n-1}(x)H(x+1) + H(x) = \phi(x). \tag{1'}$$

Then according to the theorem quoted in part II above, there exists a fundamental system of solutions,

$$H_1(x), \quad H_2(x), \ldots ., \quad H_n(x),$$

of the corresponding homogeneous equation each function of which is analytic throughout the finite plane; and these functions verify the relations

$$\lim H_j(x)\{\Gamma(x)\}^{-a}\rho_j^{-x}x^{-m_j} = 1, \qquad j = 1, \ldots ., n, \tag{5}$$

where the limit is taken for x approaching infinity in the positive sense along any line parallel to the real axis. Here the constants ρ_j, $j = 1, \ldots ., n$, are the roots of the associated algebraic equation and m_j, $j = 1, \ldots ., n$, are properly determined constants.

We seek now a solution $H^1(x)$ of (1') of the form

$$H^1(x) = \alpha_1(x)H_1(x) + \ldots . + \alpha_n(x)H_n(x).$$

Corresponding to equations (3) we have now the system

$$\begin{aligned} &H_1(x+1)\Delta\alpha_1(x)+\ldots\ldots+H_n(x+1)\Delta\alpha_n(x)=0,\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\\ &H_1(x+n-1)\Delta\alpha_1(x)+\ldots\ldots+H_n(x+n-1)\Delta\alpha_n(x)=0,\\ &H_1(x+n)\Delta\alpha_1(x)+\ldots\ldots+H_n(x+n)\Delta\alpha_n(x)=\phi(x). \end{aligned} \tag{6}$$

The determinant of this system is $D(x+1)$ where

$$D(x)=\begin{vmatrix} H_1(x) & H_2(x) & \ldots\ldots H_n(x) \\ H_1(x+1) & H_2(x+1) & \ldots\ldots H_n(x+1) \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \\ H_1(x+n-1) & H_2(x+n-1) & \ldots\ldots H_n(x+n-1) \end{vmatrix}$$

If we employ the notation $\lim \psi(x)$ in the same sense as in (5), then in view of (5) and the definition of $D(x)$ it is easy to prove that

$$\lim D(x)\{\Gamma(x)\Gamma(x+1)\ldots\ldots \Gamma(x+n-1)\}^{-a}(\rho_1\rho_2\ldots\ldots\rho_n)^{-x}x^{-m_1-m_2\ldots\ldots-m_n}=D, \tag{7}$$

where

$$D=\begin{vmatrix} 1 & 1 & \ldots\ldots 1 \\ \rho_1^{-1} & \rho_2^{-1} & \ldots\ldots \rho_n^{-1} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \rho_1^{-n+1} & \rho_2^{-n+1} & \ldots\ldots \rho_n^{-n+1} \end{vmatrix},$$

whence D is a constant different from zero, since the quantities $\rho_1, \rho_2, \ldots\ldots, \rho_n$ are all distinct.

By Stirling's formula for the gamma function we have

$$\lim \Gamma(x+r)\,(x+r)^{-x-r+\frac{1}{2}}e^{x+r}=\sqrt{2\pi},$$

where r is any integer. Hence we have readily

$$\lim \Gamma(x+r)x^{-x}e^{x}x^{-r+\frac{1}{2}}=\sqrt{2\pi}.$$

Taking this result in connection with (7) it is easy to see that there exist constants $a, m, c, c \neq 0$, such that

$$\lim D(x)x^{nax}a^{-x}x^{-m}=c. \tag{8}$$

Now $D(x)$ satisfies a difference equation of the first order which may be found in the following manner: In the determinant defining $D(x)$ replace x by $x+1$ and then substitute for $H_i(x+n)$ its value

$$H_i(x+n)=-\frac{1}{P_0(x)}\,[P_1(x)H_i(x+n-1)+\ldots\ldots+H_i(x)]$$

gotten by means of the homogeneous equation corresponding to (1). One thus obtains the result

$$D(x+1)=(-1)^n\frac{D(x)}{P_0(x)}\,.$$

If we make the transformation

$$\overline{D}(x)=D(x)\,x^{nax}a^{-x}x^{-m}$$

we have readily the result that

$$\overline{D}(x+1)=\{1+\psi(x)\}\overline{D}(x),\qquad \lim\overline{D}(x)=c, \tag{9}$$

where $\psi(x)$ is analytic at infinity and is expansible in the form

$$\psi(x)=\psi''x^{-2}+\psi'''x^{-3}+\ldots.,\qquad |x|\geqq R,$$

R being an appropriately chosen constant. That $\overline{D}(x)$ is uniquely defined by the fact that it satisfies relations (9) is easily shown. For, if $\overline{D}_1(x)$ is any function satisfying the given difference equation then the most general solution of that equation is $p(x)\overline{D}_1(x)$, where $p(x)$ has the period 1. The existence and value of the limits $\lim\overline{D}_1(x)=c$, $\lim p(x)\overline{D}_1(x)=c$ require that $p(x)$ shall equal 1. Hence $\overline{D}(x)$ is uniquely defined by relations (9).

In view of the general theorem in § 1 of my dissertation, p. 108, it is now clear that $\overline{D}(x)$ is such that there exists a line parallel to the axis of imaginaries to the right of which $\overline{D}(x)$ has no zero. Then, from the relation between $\overline{D}(x)$ and $D(x)$ it follows that there exists a line l to the right of which $D(x)$ has no zero.

From these results we conclude that system (1′) has a solution in the form

$$\Delta\alpha_i(x)=M_i(x),\qquad i=1,\ldots.,n, \tag{10}$$

where the functions $M_i(x)$ have no singularities to the right of the line l. A theorem of Hurwitz* then states that there exist functions $\alpha_i(x)$ satisfying equations (10) and having the property that there exists a line λ, parallel to the axis of imaginaries, to the right of which no one of these functions has a singularity. Hence, if we take such values of $\alpha_i(x)$, the function $H^1(x)$ has no singularities to the left of the line λ. Then, making use of the fact that $H^1(x)$ satisfies (1′), we conclude readily that $H^1(x)$ has no singularity in the finite plane. Hence the general solution of (1′) may be written in the form

$$H(x)=H^1(x)+p_1(x)H_1(x)+\ldots.+p_n(x)H_n(x),$$

where $p_1(x),\ldots.,p_n(x)$ are arbitrary periodic functions of period 1 and the functions $H^1(x)$, $H_1(x)$, …., $H_n(x)$ have no singularities in the finite plane.

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* *Acta Mathematica* Vol. XX (1897), pp. 308-310.

In a similar manner it may be shown that if we modify (1) by taking $P_o(x)$ instead of $P_n(x)$ equal to 1 and making $\phi(x)$ entire as before, we obtain the same general result.

Thus we have the following theorem:

II. *If*

$$P_0(x)H(x+n)+P_1(x)H(x+n-1)+\ldots\ldots+P_n(x)H(x)=\phi(x)$$

is a difference equation of simple character in which $\phi(x)$ *is an entire function and the coefficients* $P_i(x)$, $i=0, 1, \ldots\ldots, n$, *are polynomials in* x, *and if either* $P_0(x)\equiv 1$ *or* $P_n(x)\equiv 1$, *then the general solution* $H(x)$ *may be written in the form*

$$H(x)=E(x)+p_1(x)E_1(x)+\ldots\ldots+p_n(x)E_n(x),$$

where $E(x)$, $E_1(x)$, $\ldots\ldots$, $E_n(x)$ *are entire functions and* $p_1(x)$, $\ldots\ldots$, $p_n(x)$ *are arbitrary periodic functions of period* 1.

INDIANA UNIVERSITY, *April,* 1912.